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Representing homology classes of almost definite 4-manifolds

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Abstract

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For simply connected 4-manifolds $\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$, the representing problem of homology classes by embedded spheres is studied. A nonrepresentability condition is given. By using properties about automorphisms of unimodular quadratic forms over integers, some representability results are obtained.

Keywords: 4-manifold; Embedded sphere.

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1. Introduction

The content of this paper is organized as follows: first, using the result of Wall [5], we analyse a basic property about automorphisms of certain unimodular quadratic forms over integers. This is given mainly for later argument. Secondly, we investigate the problem about representing homology classes in almost definite 4-manifolds $\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$ and obtain some nonrepresentability results about homology classes with positive square (Theorem 3.1). Combining this with the first part, we can determine such classes completely for $m = 2$ and 3 (Theorems 3.3 and 3.4). As a consequence, we may deduce the theorems of Kuga [2] and Lawson [3] from our results. Finally, some positive results are given (Theorem 3.7).

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2. Action of orthogonal groups

Let X be a free Abelian group with a symmetric bilinear map ϕ of $X \times X$ to the integers, with determinant ± 1, write $x \cdot y$ for $\phi(x, y)$. An automorphism of the bilinear map is an automorphism of group X, such that $\phi(Tx, Ty) = \phi(x, y)$ for all $x, y \in X$. These automorphisms form a group $\mathscr{O}(\mathscr{H})$, called the orthogonal group of X.

Denote I^+ (or I^-) to be the free Abelian group of rank 1, with basis u (or v) satisfying $u \cdot u = 1$ (or -1).

Theorem 2.1. *For $n \leqslant 3$, under $\mathscr{O}(I^+ + nI^-)$, each vector of positive norm is equivalent to a reduced vector $au + b_1v_1 + \cdots + b_nv_n$ with $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$, $a \geqslant b_1 + \cdots + b_n$.*

The theorem is obvious for $n = 1$. For $n = 2$, this is [5, Theorem 2.3]. Thus we need only to prove it for $n = 3$.

Lemma 2.2 (see [5, 1.6]). *$\mathscr{O}(I^+ + 3I^-)$ is generated by trivial automorphisms and R, where trivial automorphisms refer to permutations of v_i, v_j; changing the signs of u and (or) v_i; and*

$$R: \begin{cases} u \mapsto 2u + v_1 + v_2 + v_3, \\ v_1 \mapsto -u - v_2 - v_3, \\ v_2 \mapsto -u - v_1 - v_3, \\ v_3 \mapsto -u - v_1 - v_2. \end{cases}$$

here u, v_i are the basis of I^+ and I^- respectively.

Proof of Theorem 2.1 for $n = 3$. After probably changing the sign of u and v, we may assume $au + b_1v_1 + b_2v_2 + b_3v_3$ satisfy $a, b_i \geqslant 0$. If $a < b_1 + b_2 + b_3$, then $R(au + b_1v_1 + b_2v_2 + b_3v_3) = a'u + b_1'v_1 + b_2'v_2 + b_3'v_3$, where $a' = 2a - b_1 - b_2 - b_3 < a$.

Note that $a^2 - b_1^2 - b_2^2 - b_3^2 > 0$, so

$$\begin{aligned} 4a^2 &> 4(b_1^2 + b_2^2 + b_3^2) \\ &> (b_1^2 + b_2^2 + b_3^2) + (b_1^2 + b_2^2) + (b_2^2 + b_3^2) + (b_3^2 + b_1^2) \\ &\geqslant (b_1 + b_2 + b_3)^2. \end{aligned}$$

Hence

$$2a > b_1 + b_2 + b_3.$$

Hence

$$0 < a' < a.$$

Next, we induct on a. If $|a|=1$, then $1 \geqslant b_1^2+b_2^2+b_3^2 \geqslant |b_1|+|b_2|+|b_3|$. For arbitrary a with $|a|<|b_1|+|b_2|+|b_3|$, we can use R on $|a|u+|b_1|v_1+|b_2|v_2+|b_3|v_3$ to decrease $|a|$ till we have $|a| \geqslant |b_1|+|b_2|+|b_3|$.Then use trivial automorphisms to get the desired vector. □

3. Representing homology classes

Theorem 3.1. *Let $\xi, \eta_1, \eta_2, \ldots, \eta_m$ be canonical generators of $H_2(\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2, \mathbb{Z})$. $u = a\xi + \sum_{i=1}^m b_i\eta_i$ has positive square, $a, b_i \in \mathbb{Z}$. We can suppose $|b_1| \geqslant |b_2| \geqslant \cdots \geqslant |b_m|$, w.l.o.g. If $|a| \neq |b_1|+1$ and $u^2 \geqslant \max\{\sum_{i=2}^m b_i^2+4, 5\}$, then u cannot be represented by an embedded sphere.*

Proof. Denote $n = u^2 = a^2 - \sum_{i=1}^m b_i^2$, w.l.o.g., we may assume $a > b_1 \geqslant b_2 \geqslant \cdots \geqslant b_m \geqslant 0$, then $n \geqslant \max\{\sum_{i=2}^m b_i^2+4, 5\}$. Let $M = \mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2 \# (n-1)\overline{\mathbb{C}P}^2$, with ζ_i, $i = 1, 2, \ldots, n-1$, being the generators of $H_2(M, Z)$ with respect to the additional $(n-1)\overline{\mathbb{C}P}^2$. Suppose conversely that u is represented by an embedded sphere. Then $v = u + \sum_{i=1}^{n-1}\zeta_i$ can be represented by a smoothly embedded sphere S in M. The self-intersection number of $S = S \cdot S = v^2 = u^2 - (n-1) = 1$. Hence the tubular neighborhood N of S in M is the $(+1)$–Hopf bundle over S. So ∂N is diffeomorphic to S^3. Set $W^4 = (M^4 - \text{int } N) \cup_\partial D^4$. It is known [2] that W^4 is a closed, simply connected 4-manifold with negative intersection form. By Donaldson's result [1], the intersection form of W is standard. But $M = W \# \tilde{N}$, where $\tilde{N} = N \cup_\partial D^4 \cong \mathbb{C}P^2$. Hence the intersection form of M is the direct sum of that of W and $\tilde{N}$. Therefore $(H_2(W), \langle,\rangle_W) \cong (v^\perp, \langle,\rangle_M)$. So there are $2(m+n-1)$ α's satisfying $\alpha \in H_2(W, Z)$, $\alpha \cdot v = 0$ and $\alpha^2 = -1$.

Write $\alpha = x\xi + \sum_{i=1}^m y_i\eta_i + \sum_{j=1}^{n-1} z_j\zeta_j$. Then the Diophantine equations

$$\begin{cases} ax = \sum\limits_{i=1}^{m} b_i y_i + \sum\limits_{j=1}^{n-1} z_j, \\ x^2 + 1 = \sum\limits_{i=1}^{m} y_i^2 + \sum\limits_{j=1}^{n-1} z_j^2 \end{cases} \qquad (*)$$

shall have $2(m+n-1)$ solutions.

$$b_1^2 y_1^2 = \left(ax - \sum_{i=2}^{m} b_i y_i - \sum_{j=1}^{n-1} z_j \right)^2$$

$$= b_1^2 \left(x^2 + 1 - \sum_{i=2}^{m} y_i^2 - \sum_{j=1}^{n-1} z_j^2 \right),$$

i.e.,

$$\left(a^2-b_1^2\right)x^2-2a\left(\sum_{i=2}^{m} b_i y_i+\sum_{j=1}^{n-1} z_j\right)x+\left(\sum_{i=2}^{m} b_i y_i+\sum_{j=1}^{n-1} z_j\right)^2$$
$$+b_1^2\left(\sum_{i=2}^{m} y_i^2+\sum_{j=1}^{n-1} z_j^2-1\right)=0.$$

We view it as a quadratic equation of x, and denote Δ to be its discriminant. Then

$$\tfrac{1}{4}\Delta=a^2\left(\sum_{i=2}^{m} b_i y_i+\sum_{j=1}^{n-1} z_j\right)^2-\left(a^2-b^2\right)$$
$$\times\left[\left(\sum_{i=2}^{m} b_i y_i+\sum_{j=1}^{n-1} z_j\right)^2+b_1^2\left(\sum_{i=2}^{m} y_i^2+\sum_{j=1}^{n-1} z_j^2-1\right)\right]$$
$$=b_1^2\left(\sum_{i=2}^{m} b_i y_i+\sum_{j=1}^{n-1} z_j\right)^2-b_1^2\left(a^2-b_1^2\right)\left(\sum_{i=2}^{m} y_i^2+\sum_{j=1}^{n-1} z_j^2-1\right).$$

Suppose $b_1\neq 0$ and let

$$d=\frac{1}{4b_1^2}\Delta=\left(\sum_{i=2}^{m} b_i y_i+\sum_{j=1}^{n-1} z_j\right)^2-\left(a^2-b_1^2\right)\left(\sum_{i=2}^{m} y_1^2+\sum_{j=1}^{n-1} z_j^2-1\right).$$

Denote r to be the number of z_j's which are nonzero, s to be the number of z_j's with $|z_j|>1$. Then $0\leqslant s\leqslant r\leqslant n-1$. After renumbering, we may suppose $z_j\neq 0$ for $1\leqslant j\leqslant r$, and $|z_j|>1$, for $1\leqslant j\leqslant s$. We divide equation $(*)$ into four cases.

Case A: $r=n-1$. Then,

$$d=\left(\sum_{i=2}^{m} b_i y_i+\sum_{j=1}^{n-1} z_j\right)^2-\left(a^2-b_1^2\right)\left(\sum_{i=2}^{m} y_i^2+\sum_{j=1}^{n-1} z_j^2-1\right)$$
$$\leqslant\left(\sum_{i=2}^{m} b_i|y_i|+\sum_{j=1}^{s}|z_j|+n-s-1\right)^2$$
$$-\left(a^2-b_1^2\right)\left(\sum_{i=2}^{m} y_i^2+\sum_{j=1}^{s} z_j^2+n-s-2\right)$$

(here we use the definition of s and the condition $r=n-1$)

$$=\left(\sum_{i=2}^{m} b_i|y_i|+\sum_{j=1}^{s}|z_i|\right)^2+2(n-s-1)\left(\sum_{i=2}^{m} b_i|y_i|+\sum_{j=1}^{s}|z_j|\right)$$
$$+(n-s-1)^2-\left(a^2-b_1^2\right)\left(\sum_{i=2}^{m} y_i^2+\sum_{j=1}^{s} z_j^2\right)$$

$$
\begin{aligned}
&-(a^2-b_1^2)(n-s-2)\\
&\leqslant \left(s+\sum_{i=2}^m b_i^2\right)\left(\sum_{i=2}^m y_i^2+\sum_{j=1}^s z_j^2\right)+2(n-s-1)\left(\sum_{i=2}^m b_i|y_i|+\sum_{j=1}^s |z_j|\right)\\
&\quad+(n-s-1)^2-\left(n+\sum_{i=2}^m b_i^2\right)\left(\sum_{i=2}^m y_i^2+\sum_{j=1}^s z_j^2\right)\\
&\quad-\left(n+\sum_{i=2}^m b_i^2\right)(n-s-2)
\end{aligned}
$$

(here the Cauchy inequality was used for the first term)

$$
\begin{aligned}
&=(s-n)\left(\sum_{i=2}^m y_i^2+\sum_{j=1}^s z_j^2\right)+2(n-s-1)\left(\sum_{i=2}^m b_i|y_i|+\sum_{j=1}^s |z_j|\right)\\
&\quad-\left(s+\sum_{i=2}^m b_i^2\right)n+(s+1)^2+(s+2)\sum_{i=2}^m b_i^2\\
&=\left((s-n)\sum_{j=1}^s z_j^2+2(n-s-1)\sum_{j=1}^s |z_j|\right)\\
&\quad+\left((s-n)\sum_{i=2}^m y_i^2+2(n-s-1)\sum_{i=2}^m b_i|y_i|-n\sum_{i=2}^m b_i^2\right)\\
&\quad-sn+(s+1)^2+(s+2)\sum_{i=2}^m b_i^2.
\end{aligned}
$$

Note that $|z_j|\geqslant 2$, so $2(n-s-1)\Sigma_{j=1}^s|z_j|\leqslant(n-s-1)\Sigma_{j=1}^s z_j^2$, thus

$$
\begin{aligned}
d&\leqslant -\sum_{j=1}^s z_j^2-(n-s-1)\sum_{i=2}^m(|y_i|-b_i)^2-\sum_{i=2}^m y_i^2-(s+1)\sum_{i=2}^m b_i^2-sn\\
&\quad+(s+1)^2+(s+2)\sum_{i=2}^m b_i^2\\
&\leqslant -sn-(n-s-1)\sum_{i=2}^m(|y_i|-b_i)^2-\sum_{i=2}^m y_i^2-4s+(s+1)^2+\sum_{i=2}^m b_i^2\\
&= -sn+(s-1)^2+\sum_{i=2}^m b_i^2-(n-s-1)\sum_{i=2}^m(|y_i|-b_i)^2-\sum_{i=2}^m y_i^2.
\end{aligned}
$$

(i) If $s\geqslant\Sigma_{i=2}^m b_i^2\neq 0$ (we assume $s\geqslant 1$ if $\Sigma_{i=2}^m b_i^2=0$), then

$$
\begin{aligned}
d&\leqslant -sn+(s-1)^2+\sum_{i=2}^m b_i^2\\
&< -s(s+1)+s^2+\sum_{i=2}^m b_i^2\\
&\leqslant 0.
\end{aligned}
$$

(ii) If $1 \leqslant s < \sum_{i=2}^{m} b_i^2$, then

$$\begin{aligned} d &\leqslant -sn + (s-1)^2 + \sum_{i=2}^{m} b_i^2 \\ &< -sn + s^2 + \sum_{i=2}^{m} b_i^2 \\ &\leqslant \max\left\{ -n + 1 + \sum_{i=2}^{m} b_i^2,\ -\left(\sum_{i=2}^{m} b_i^2\right)n + \left(\sum_{i=2}^{m} b_i^2\right)^2 + \sum_{i=2}^{m} b_i^2 \right\} \\ &< 0 \end{aligned}$$

since n $\geqslant \sum_{i=2}^{m} b_i^2 + 4$.

(iii) If $s = 0$, and there exists $l \in \{2, \dots, m\}$, such that $|y_l| \neq b_l$, then

$$\begin{aligned} d &\leqslant -(n-1) \sum_{i=2}^{m} (|y_i| - b_i)^2 - \sum_{i=2}^{m} y_i^2 + 1 + \sum_{i=2}^{m} b_i^2 \\ &\leqslant -(n-1) + 1 + \sum_{i=2}^{m} b_i^2 \\ &< 0. \end{aligned}$$

(iv) If $s = 0$ and $|y_i| = b_i$ for all $i = 2, \dots, m$, we have $d \leqslant 1$. The equality holds iff there are no different signs among y_i, z_j for all $i = 2, \dots, m$, $j = 1, \dots, n-1$. Then the Diophantine equations $(*)$ become

$$\begin{cases} x^2 + 1 = \sum_{i=2}^{m} b_i^2 + n - 1 + y_1^2 = a^2 - b_1^2 - 1 + y_1^2, \\ ax = \pm\left(\sum_{i=2}^{m} b_i^2 + n - 1\right) + b_1 y_1 = \pm\left(a^2 - b_1^2 - 1\right) + b_1 y_1. \end{cases}$$

When $a \neq b_1 + 1$, eliminating y_1 we get

$$\left(a^2 - b_1^2\right)x^2 \pm 2a\left(a^2 - b_1^2 - 1\right)x + \left(a^2 - b_1^2 - 1\right)^2 + b_1^2\left(a^2 - b_1^2 - 2\right) = 0.$$

So

$$x = \pm a \pm \frac{a \pm b_1}{a^2 - b_1^2},$$

which is not an integer. Hence there is no solution when $a \neq b_1 + 1$.

When $a = b_1 + 1$, we have two solutions

$$\begin{cases} x = b_1, \\ y_1 = b_1 - 1, \\ y_i = b_i, \\ z_j = 1, \text{ and} \end{cases} \qquad \begin{cases} x = -b_1, \\ y_1 = -b_1 + 1, \\ y_i = -b_i, \\ z_j = -1. \end{cases}$$

If there are different signs among $\{y_i, z_j | i = 2, \ldots, m,\ j = 1, \ldots, n-1\}$, it is easy to check that $d < 0$.

Case B: $2 \leqslant r \leqslant n-2$. Then

$$d = \left(\sum_{i=2}^{m} b_i y_i + \sum_{j=1}^{r} z_j\right)^2 - \left(a^2 - b_1^2\right)\left(\sum_{i=2}^{m} y_i^2 + \sum_{j=1}^{n-1} z_j^2 - 1\right)$$

$$\leqslant \left(r + \sum_{i=2}^{m} b_i^2\right)\left(\sum_{i=2}^{m} y_i^2 + \sum_{j=1}^{r} z_j^2\right) - \left(n + \sum_{i=2}^{m} b_i^2\right)\left(\sum_{i=2}^{m} y_i^2 + \sum_{j=1}^{n-1} z_j^2 - 1\right)$$

(here the Cauchy inequality was used and equality holds iff $b_i/y_i = z_j$ for all i, j)

$$= (r-n)\left(\sum_{i=2}^{m} y_i^2 + \sum_{j=1}^{r} z_j^2\right) + n + \sum_{i=2}^{m} b_i^2$$

$$< r(r-n) + n + \sum_{i=2}^{m} b_i^2.$$

Since $2 \leqslant r \leqslant n-2$, $r(r-n)$ reaches its maximum exactly when $r = 2$ or $n-2$. So

$$d < 2(2-n) + n + \sum_{i=2}^{m} b_i^2$$

$$\leqslant 0,$$

since $n \geqslant \max\{\sum_{i=2}^{m} b_i^2 + 4, 5\}$.

Case C: $r = 1$. Then

$$d = \left(\sum_{i=2}^{m} b_i y_i + z_l\right)^2 - \left(a^2 - b_1^2\right)\left(\sum_{i=2}^{m} y_i^2 + z_l^2 - 1\right)$$

$$\leqslant \left(1 + \sum_{i=2}^{m} b_i^2\right)\left(\sum_{i=2}^{m} y_i^2 + z_l^2\right) - \left(n + \sum_{i=2}^{m} b_i^2\right)\left(\sum_{i=2}^{m} y_i^2 + z_l^2 - 1\right)$$

$$= -(n-1)\left(\sum_{i=2}^{m} y_i^2 + z_l^2\right) + n + \sum_{i=2}^{m} b_i^2.$$

(i) If there exists some i such that $y_i \neq 0$ or $|z_l| \neq 1$, then $d < -2(n-1) + n + \sum_{i=2}^{m} b_i^2 < 0$.

(ii) If $y_i = 0$ for all $i = 2, \ldots, m$, and $|z_l| = 1$, then equation $(*)$ becomes

$$\begin{cases} ax = b_1 y_1 + z_l, \\ x^2 + 1 = y_1^2 + 1, \\ z_l = \pm 1, \end{cases}$$

which has no solutions when $a \neq b_1 + 1$. When $a = b_1 + 1$, we solve it

$$\begin{cases} x = y_1 = z_l = \pm 1, \\ a = b_1 + 1, \\ y_i = 0,\ i = 2, \ldots, m, \end{cases}$$

where l may be anyone that belongs to $\{1, \ldots, n-1\}$, which has $2(n-1)$ solutions.

Case D: $r = 0$. Then

$$\begin{aligned} d &= \left(\sum_{i=2}^{m} b_i y_i \right)^2 - \left(a^2 - b_1^2\right)\left(\sum_{i=2}^{m} y_i^2 - 1 \right) \\ &\leqslant \left(\sum_{i=2}^{m} b_i^2 \right)\left(\sum_{i=2}^{m} y_i^2 \right) - \left(n + \sum_{i=2}^{m} b_i^2 \right)\left(\sum_{i=2}^{m} y_i^2 - 1 \right) \\ &= -n \sum_{i=2}^{m} y_i^2 + n + \sum_{i=2}^{m} b_i^2. \end{aligned}$$

(i) If $\sum_{i=2}^{m} y_i^2 \geqslant 2$, then

$$d \leqslant -2n + n + \sum_{i=2}^{m} b_i^2 < 0.$$

(ii) If $\sum_{i=2}^{m} y_i^2 = 1$, i.e., there is some $l \in \{2, \ldots, m\}$ such that $|y_l| = 1$ and $y_i = 0$ for all $i \neq l$. Then equation $(*)$ becomes

$$\begin{cases} x^2 + 1 = y_1^2 + 1, \\ ax = b_1 y_1 + b_l y_l, \\ y_l = \pm 1. \end{cases}$$

That is

$$\begin{cases} x = y_1,\ z_j = 0,\ j = 1, \ldots, n-1, \\ y_l = \pm 1,\ l \in \{2, \ldots, m\}, \\ y_i = 1,\ i \in \{2, \ldots, m\} \text{ and } i \neq l, \\ (a - b_1)x = b_l y_l, \end{cases}$$

which has at most $2(m-1)$ solutions.

(iii) If $\sum_{i=2}^{m} y_i^2 = 0$, i.e., $y_i = 0$ for $i = 2, \ldots, m$, then $(*)$ becomes

$$\begin{cases} x^2 + 1 = y_1^2, \\ ax = b_1 y_1, \end{cases}$$

which has no solutions.

Summarizing the above deductions, we see that if $a \neq b_1 + 1$ and $n \geqslant \sum_{i=2}^{m} b_i^2 + 4$ or 5, then the Diophantine equations $(*)$ have fewer than $2(n + m - 1)$ solutions.

The above discussions are based on $b_1 \neq 0$. If $b_1 = 0$, this means $b_i = 0$ for all $i = 1, \ldots, m$, then $(*)$ becomes

$$\begin{cases} ax = \sum_{j=1}^{n-1} z_j, \\ x^2 + 1 = \sum_{i=2}^{m} y_i^2 + \sum_{j=1}^{n-1} z_j^2, \ n = a^2. \end{cases}$$

So, we use the Cauchy inequality

$$a^2x^2 = \left(\sum_{j=1}^{r} z_j \right)^2 \leqslant r \sum_{j=1}^{r} z_j^2 \leqslant r(x^2 + 1).$$

Hence

$$x^2 \leqslant r,$$

$$\sum_{i=1}^{m} y_i^2 + \sum_{j=1}^{r} z_j^2 \leqslant r + 1.$$

This implies $|z_j| = 1$ for $j = 1, \ldots, r$ and $\sum_{i=1}^{m} y_i^2 \leqslant 1$.

If $\sum_{i=1}^{m} y_i^2 = 1$, then

$$\begin{cases} x^2 + 1 = 1 + \sum_{j=1}^{r} z_j^2 = 1 + r, \\ ax = r - 2r', \end{cases}$$

where r' is the number of z_j's with $z_j = -1$.

$$a^2x^2 = (r - 2r')^2,$$

so

$$a^2r = (r - 2r')^2 \leqslant r^2 \leqslant (a^2 - 1)r.$$

Therefore

$$r = 0.$$

In this case there are $2m$ solutions:

$$\begin{cases} x = z_j = 0, & j = 1, \ldots, n-1, \\ y_l = \pm 1, & l \in \{1, \ldots, m\}, \\ y_i = 0, & \text{for all } i \neq l. \end{cases}$$

If $\sum_{i=1}^{m} y_i^2 = 0$, then

$$ax = r - 2r',$$

$$x^2 + 1 = r,$$

$$a^2x^2 = (r - 2r')^2 = a^2(r - 1).$$

Note $r \leqslant n-1 = a^2-1$, so we have $r'=0$. Therefore,

$$x^2+1 = ax = r,$$
$$x^2 - ax + 1 = 0,$$

which has integer solutions iff $a = \pm 2$.

Thus the theorem is proved. $\square$

From the proof, we can also see

Theorem 3.2. *$u = a\xi \in H_2(\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2, \mathbb{Z})$ is represented by an embedded sphere iff $|a| \leqslant 2$, where ξ is the canonical generator of $H_2(\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2, \mathbb{Z})$ with respect to $\mathbb{C}P^2$.*

In Section 2, we proved for $m \leqslant 3$, each element in $H_2(\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2, \mathbb{Z})$ with positive square is equivalent under some automorphisms to a reduced vector $au + b_1 v_1 + \cdots + b_m v_m$ with $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_m \geqslant 0$ and $a \geqslant b_1 + b_2 + \cdots + b_m$. Theorem 2 of Wall [6] says every such automorphism is induced by a diffeomorphism of M. Thus for $n=2$ or 3, we need only to consider such reduced vectors and this yields:

Theorem 3.3. *Let $u = (a, b_1, b_2) \in H_2(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2, \mathbb{Z})$ satisfy $|b_1| \geqslant |b_2|$ and $|a| \geqslant |b_1| + |b_2|$. Then u is represented by an embedded sphere iff $|a| - |b_1| \leqslant 1$ or $u = (\pm 2, 0, 0)$.*

Theorem 3.4. *Let $u = (a, b_1, b_2, b_3) \in H_2(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2, \mathbb{Z})$ satisfy $|b_1| \geqslant |b_2| \geqslant |b_3|$ and $|a| \geqslant |b_1| + |b_2| + |b_3|$. Then u is represented by an embedded sphere iff $b_3 = 0$ and $|a| - |b_1| \leqslant 1$ or $u = (\pm 2, 0, 0, 0)$.*

The if part is known (see Wall [6] or Lawson [3]). The other part of Theorems 3.3 and 3.4 is direct from Theorem 3.2.

As a consequence we can deduce Lawson's theorem on representing homology classes in $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and that of Kuga on $S^2 \times S^2$ from our results.

Corollary 3.5 (Lawson and Luo). *$u = ax + by \in H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \mathbb{Z})$ is represented by an embedded sphere iff $||a| - |b|| \leqslant 1$ or $(a, b) = (0, \pm 2)$ or $(\pm 2, 0)$.*

Proof. The if part is known (see Wall [6]). Suppose $||a| - |b|| \geqslant 2$, w.l.o.g. we may suppose $a - 2 \geqslant b \geqslant 0$. If (a, b) is represented by an embedded sphere in $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, then $(a, b, 1)$ is represented in $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2$, which contradicts to Theorem 3.3, since $a \geqslant b + 2$. $\square$

Corollary 3.6 (Kuga). *$p\xi + q\eta \in H_2(S^2 \times S^2, \mathbb{Z})$ is represented by an embedded sphere iff $|p| \leqslant 1$ or $|q| \leqslant 1$.*

Proof. The if part is trivial. Suppose w.l.o.g. $p \geqslant 2$, $q \geqslant 2$ and $p\xi + q\eta$ is represented by an embedded sphere. Then $p\xi + q\eta + \zeta$ is represented in $S^2 \times S^2 \# \overline{\mathbb{C}P}^2$, where ζ is the canonical generator of $H_2(\overline{\mathbb{C}P}^2)$. Note $S^2 \times S^2 \# \overline{\mathbb{C}P}^2$ is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2$. Let $\xi = u - v_1$, $\eta = u - v_2$, $\zeta = v_1 + v_2 - u$. Then $u^2 = -v_1^2 = -v_2^2 = 1$, $uv_1 = uv_2 = v_1 v_2 = 0$. Hence u, v_1, v_2 can be viewed as obtained from the canonical generators of $H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2, \mathbb{Z})$ through an automorphism. $p\xi + q\eta + \zeta = (p + q - 1)u - (p - 1)v_1 - (q - 1)v_2$. By Theorem 3.3, it is not represented by a smoothly embedded sphere. A contradiction. $\square$

Note in $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, if $||a| - |b|| = 1$, then $a\xi + b\eta \in H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \mathbb{Z})$ is represented by an embedded sphere, where ξ, η are the standard generators of $H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \mathbb{Z})$. The next theorem shows a similar result.

Theorem 3.7. *Let $u = a\xi + \sum_{i=1}^m b_i \eta_i \in H_2(\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2, \mathbb{Z})$, where ξ, η_i are the canonical generators of $H_2(\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2, \mathbb{Z})$. If $||a| - |b_i|| = 1$ for some $i \in \{1, \ldots, m\}$, then u can be represented by an embedded sphere.*

Proof. We induct on m. For $m = 2$, w.l.o.g. we may assume $||a| - |b_1|| = 1$. Change the sign of generators if necessary, we can make a, b_1, b_2 positive. Then $a = b_1 + k$, $k = 1$ or -1. It is easy to check that

$$T_1 : \begin{cases} \xi \to 3\xi + 2\eta_1 + 2\eta_2, \\ \eta_1 \to -2\xi - \eta_1 - 2\eta_2, \\ \eta_2 \to -2\xi - 2\eta_1 - \eta_2 \end{cases}$$

is an automorphism of $H_2(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2, \mathbb{Z})$.

$$\begin{aligned} (b_1 + k, b_1, b_2) &\xrightarrow{T_1} (b_1 + 3k - 2b_2, b_1 + 2k - 2b_2, 2k - b_2) \\ &\xrightarrow{T_2} (b_1 + 3k - 2b_2, b_1 + 2k - 2b_2, b_2 - 2k) \end{aligned}$$

where T_2 is the automorphism changing of the sign of η_2. Set $T = T_2 \cdot T_1$, then

$$(b_1 + k, b_1, b_2) \xrightarrow{T} \left(b_1^{(1)} + k, b_1^{(1)}, b_2^{(1)}\right)$$

where $b_1^{(1)} = b_1 + 2k - 2b_2$, $b_2^{(1)} = b_2 - 2k$.

Denote $b_i^{(n)} = T^n b_i$, $i = 1, 2$. So we have,

$$\begin{cases} b_2^{(n+1)} = b_2^{(n)} - 2k, \\ b_1^{(n+1)} = b_1^{(n)} + 2k - 2b_2^{(n)}. \end{cases}$$

Hence

$$\begin{cases} b_2^{(n)} = b_2 - 2kn, \\ b_1^{(n)} = b_1 - 2b_2 n + 2kn^2. \end{cases}$$

We can choose n, such that $0 \leqslant b_2 - 2n \leqslant 1$.

If $k=1$, then $(b_1+1, b_1, b_2) \xrightarrow{T^n} (b_1^{(n)}+1, b_1^{(n)}, b_2^{(n)})$ which is obviously represented by an embedded sphere, since $b_2^{(n)}=0$ or 1. By Theorem 2 of Wall [6], $(b_1+1, b_1, b_2)=(a, b_1, b_2)$ is represented by a smoothly embedded sphere. If $k=-1$, set $T'=T_1 \circ T_2$, then $(b_1-1, b_1, b_2) \xrightarrow{T'^n} (b_1^{(n)}-1, b_1^{(n)}, b_2^{(n)})$, where $b_i^{(n)}=T'^n b_i$ and $b_2^{(n)}=b_2+2kn=b_2-2n=0$ or 1. Hence $(b_1-1, b_1, b_2)=(a, b_1, b_2)$ is represented by an S^2. Thus the theorem holds for $m=2$. Suppose the theorem holds for $m(\geqslant 2)$. For $m+1$, we suppose $||a|-|b_1||=1$ and a, b_1, b_2 positive. From the deduction above, we see that

$$(a, b_1, b_2, b_3, \ldots, b_{m+1}) \xrightarrow{R^n \oplus \mathrm{Id}} \left(a^{(n)}, b_1^{(n)}, b_2^{(n)}, b_3, \ldots, b_{m+1}\right)$$

where $R=T$ or T' depending on $a-b_1=1$ or -1 respectively, n is a positive integer satisfying $0 \leqslant b_2-2n \leqslant 1$. Since $|a^{(n)}-b_1^{(n)}|=1$, by the inductive hypothesis, $(a^{(n)}, b_1^{(n)}, b_3, \ldots, b_{m+1})$ is representable in $\mathbb{C}P^2\#m\overline{\mathbb{C}P}^2$. Note $b_2^{(n)}=0$ or 1, therefore $(a^{(n)}, b_1^{(n)}, b_2^{(n)}, b_3, \ldots, b_{m+1})$ is representable in $\mathbb{C}P^2\#(m+1)\overline{\mathbb{C}P}^2$ by an embedded S^2. By Theorem 2 of Wall [6] the automorphism $R^n: H_2(\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2, \mathbb{Z}) \to H_2(\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2, \mathbb{Z})$ is induced by a diffeomorphism $f: \mathbb{C}P^2\#2\overline{\mathbb{C}P}^2 \to \mathbb{C}P^2\#2\overline{\mathbb{C}P}^2$. Then, $R^n \oplus \mathrm{Id}$ is induced by the diffeomorphism $f\#\mathrm{Id}: \mathbb{C}P^2\#2\overline{\mathbb{C}P}^2\#(m-1)\overline{\mathbb{C}P}^2 \to \mathbb{C}P^2\#2\overline{\mathbb{C}P}^2\#(m-1)\overline{\mathbb{C}P}^2$, where Id is the identity map of $(m-1)\overline{\mathbb{C}P}^2$ onto itself. This proves $(a, b_1, b_2, b_3, \ldots, b_{m+1})$ is represented by an embedded sphere, and hence the theorem. □

Remark 3.8. Using a similar technique we can show that a vector $(a+2, a, b) \in H_2(\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2, \mathbb{Z})$ with positive square is represented by an embedded S^2 iff $a=n^2$, $b=2n$ or $a=4n^2+2n$, $b=4n+1$.

Remark 3.9. After submitting this paper the author became aware that a similar result was also obtained by Mr. K. Kikuchi and D.Y. Gan.

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